

**Mathematics**  
**Higher level**  
**Paper 3 – calculus**

Wednesday 18 November 2015 (afternoon)

1 hour

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**Instructions to candidates**

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 5]

The function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is defined as  $f: x \rightarrow \begin{cases} 1, & x < 0 \\ 1 - x, & x \geq 0 \end{cases}$ .

By considering limits, prove that  $f$  is

(a) continuous at  $x = 0$ ; [2]

(b) not differentiable at  $x = 0$ . [3]

2. [Maximum mark: 10]

Let  $f(x) = e^x \sin x$ .

(a) Show that  $f''(x) = 2(f'(x) - f(x))$ . [4]

(b) By further differentiation of the result in part (a), find the Maclaurin expansion of  $f(x)$ , as far as the term in  $x^5$ . [6]

3. [Maximum mark: 11]

(a) Prove by induction that  $n! > 3^n$ , for  $n \geq 7, n \in \mathbb{Z}$ . [5]

(b) Hence use the comparison test to prove that the series  $\sum_{r=1}^{\infty} \frac{2^r}{r!}$  converges. [6]

4. [Maximum mark: 14]

Consider the function  $f(x) = \frac{1}{1+x^2}$ ,  $x \in \mathbb{R}$ .

(a) Illustrate graphically the inequality,  $\frac{1}{5} \sum_{r=1}^5 f\left(\frac{r}{5}\right) < \int_0^1 f(x) dx < \frac{1}{5} \sum_{r=0}^4 f\left(\frac{r}{5}\right)$ . [3]

(b) Use the inequality in part (a) to find a lower and upper bound for  $\pi$ . [5]

(c) Show that  $\sum_{r=0}^{n-1} (-1)^r x^{2r} = \frac{1 + (-1)^{n-1} x^{2n}}{1 + x^2}$ . [2]

(d) Hence show that  $\pi = 4 \left( \sum_{r=0}^{n-1} \frac{(-1)^r}{2r+1} - (-1)^{n-1} \int_0^1 \frac{x^{2n}}{1+x^2} dx \right)$ . [4]

5. [Maximum mark: 20]

The curves  $y = f(x)$  and  $y = g(x)$  both pass through the point  $(1, 0)$  and are defined by the differential equations  $\frac{dy}{dx} = x - y^2$  and  $\frac{dy}{dx} = y - x^2$  respectively.

(a) Show that the tangent to the curve  $y = f(x)$  at the point  $(1, 0)$  is normal to the curve  $y = g(x)$  at the point  $(1, 0)$ . [2]

(b) Find  $g(x)$ . [6]

(c) Use Euler's method with steps of 0.2 to estimate  $f(2)$  to 5 decimal places. [5]

(d) Explain why  $y = f(x)$  cannot cross the isocline  $x - y^2 = 0$ , for  $x > 1$ . [3]

(e) (i) Sketch the isoclines  $x - y^2 = -2, 0, 1$ .

(ii) On the same set of axes, sketch the graph of  $f$ . [4]